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THERMOMECHANICAL CONSTITUTIVE EQUATIONS FOR SHAPE MEMORY ALLOYS

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Abstract - Shape Memory Alloys present a large variety of behaviour in function of the thermomechanical loading pathes and the microstructural states of the material. These responses are due to different physical mechanisms of deformation which are associated to the thermoelastic martensitic transformation :

- oriented growth of the martensitic plates by the applied stress in Superelasticity ;
- mobility of the interfaces between the variants of martensite in the so-called Shape Memory Effect ;
- capability of the internal stress field produced in the material by oriented defects left by some previous transformation sequences, to influence the growing of the martensite in the Two-Way Shape Memory Effect.

The determination of the constitutive equations for the mechanical behaviour of these alloys must take into account these particular mechanisms of deformation. For each physical mechanism it is necessary, at first, to make a kinematical study of the strain associated to it. After this step, an energy balance between the driving and the resistive forces is established in each case from the analysis of the Gibb's free energy of the transformation or by using the Eshelby formalism of energy momentum. Phenomenological flow rules are then determined from the classical concept of normality rule.

In this contribution, the micromechanical aspects of the phase transformation mechanisms are presented both from the statical and the kinematical point of view. Special attention is given to the internal stress state associated with variant and grain interactions.

1. - Introduction.

Complex stress-strain-temperature behaviour are observed in Shape Memory Alloys (SMA) leading to different effects :

- pseudoelasticity,
- one way shape memory effect,
- two way shape memory effect after a training sequence.

A large number of experimental observations [1] [2] [3] [4] [5] have shown that the basic mechanisms for these effects are related to a reversible austenite-martensite phase transition and to the possibility of reorientation of the martensite by twinning.

Basically, the displacive martensitic transformation is connected with a transformation strain (practically a pure shear) and an habit plane corresponding to the interface between the martensite and the austenite. Due to the symmetry of the parent phase, a complex microstructure, observed at the mesoscopic level, consists of several variants of martensite.

For the pseudoelastic and the two way shape memory effects, the observed overall behaviour belongs to the class of non linear thermoelastic materials with hysteresis.

The one way shape memory effect has some strong similarities with classical plasticity of metals but in addition a saturation strain is observed. During the last decade an important effort has been developed in order to describe this complex overall behaviour. Three class of models are proposed :

the models of Fremond [6], Lexcellent and Vacher [7], Tanaka and all [8], Fischer [9], Falk [10] are within the framework of continuum mechanics with internal variables. On the other hand, the approach of Achenbach and Muller [11], by using the methods of statistical mechanics applied to three equilibrium configurations for the variants (the lattice particle), obtains constitutive equations for the pseudoelastic and the one way shape memory behaviour.

Some important experimental and theoretical thermodynamical investigations were done by Ortin and Planes [12] [13]. They are able to separate in the external work, the part of energy which is used for reversible transformation, the part which is stored in solid via internal stress, and the part which is dissipated by dry or viscous friction.

In a more recent approach, Patoor and all [14] [15] have used micromechanical methods to introduce an interaction matrix between variant and the interactions between grains for describing the overall pseudoelastic behaviour.

The aim of this paper is to develop some micromechanical aspects related with pure martensitic phase transition in the case of SMA ; note that here some aspects related with classical plasticity are neglected. The basic idea is to assume that the transformation strain is uniform inside each variant and that the progression of transformation is related to the displacement of the interface between the two phase or between two variants of martensite.

In the first part, related to macrohomogeneous samples, we derive relations between the overall strain increment, the transformation strain field and the displacement of interfaces.

Then, using the Green function technique the internal stress field for a given transformation strain field and boundary conditions is obtained. Basic relations for the jump of stress field at interfaces are reviewed and elastic energy is calculated. If non local effects are neglected, some explicit results are proposed.

The last part deals with the thermodynamical driving force on interfaces obtained by the energy momentum tensor introduced by Eshelby [16]. This force may be used both for a micromechanical description of the overall behaviour and for phenomenological approaches.

2. - Kinematical aspects of martensitic transformation

Let the reference configuration of the solid with volume V be the austenitic polycrystal in its natural state (free stress : $\sigma = 0$) at temperature $T > A_f$. Due to some applied stress Σ and (or) cooling at a temperature $T < M_S$ some parts V^I of V are transformed into martensite. Each part V^I undergoes a uniform transformation strain ε^{TI} and internal stress are builded due to the non integrability of the field $\varepsilon^T(r)$.

The total deformation field ε (r) results from :

- an elastic part ε^e (r)
- a thermal part ε^{th} (r)
- a transformation part ε^T (r)

so that, for the displacement field u_i (r), one obtain

$$u_{(i,j)}(r) = \varepsilon_{ij}(r) = \varepsilon^{e}_{ij}(r) + \varepsilon^{th}_{ij}(r) + \varepsilon^{T}_{ij}(r) \qquad (1)$$

where (i,j) means symmetrisation of the gradient of the displacement.

By integration over the whole sample and the assumption that the volume V is macrohomogeneous, the overall deformation E is obtained by

$$E_{ij} = \frac{1}{V}\int_V \varepsilon_{ij}(r)\, dV \qquad (2)$$

If the elastic compliances s (r) ($\varepsilon^e = S\,\sigma$) and the thermal dilatations α (r) ($\varepsilon^{th}(r) = \alpha(r)\,\theta$) can be assumed homogeneous, Eq. (2) leads to

$$E_{ij} = E^{e}_{ij} + E^{th}_{ij} + E^{T}_{ij} \qquad (3)$$

where E^e and E^{th} are respectively the macroscopic elastic and thermal parts of the overall deformation.

E^{T}_{ij} is the global transformation strain due to the formation of martensite.

Since the field $\varepsilon^T(r)$ is piece wise uniform, it can be represented by a set of indicator functions $\theta^I(r)$ so that

$$\varepsilon_{ij}^{T}(r) = \sum_{I} \varepsilon_{ij}^{TI}\, \theta^{I}(r) \tag{4}$$

where ε^{TI} is uniform in the volume V_I and $\theta^{I}(r) = \begin{cases} 1 \text{ if } r \in V_I \\ 0 \text{ if } r \notin V_I \end{cases}$

The global transformation strain E^T is then obtained by

$$E_{ij}^{T} = \frac{1}{V}\sum_{I} \int_{V} \varepsilon_{ij}^{T}\, \theta^{I}(r)\, dV = \frac{1}{V}\sum_{I} \int_{V_I} \varepsilon_{ij}^{T}\, dV \tag{5}$$

Now, we are looking for the evolution δE_{ij}^{T} of the overall strain due to change δV_I of each part V^I. By derivation, one obtains

$$\delta E_{ij}^{T} = \delta\left(\frac{1}{V}\sum_{I} \int_{V_I} \varepsilon_{ij}^{TI}\, dV\right) \tag{6}$$

Due to physical arguments, we assume that ε^{TI} is constant (in time) and uniform (in space) inside each volume V^I so that :

$$\delta\, \varepsilon_{ij}^{TI} = 0 \quad (\text{in } V^I)$$

and with respect to the change δV^I :

$$\delta E_{ij}^{T} = \frac{1}{V}\sum_{I} \int_{S_I} \varepsilon_{ij}^{TI}\, \delta\xi_n\, dS_n \qquad \left(\text{with } \frac{\delta V}{V^2} \text{ negligible}\right) \tag{7}$$

where $\delta\xi_n$ is the displacement of the interface S_I associated with the elementary area-vector $dS_n = n_n\, dS$. An equivalent form of equation (7) may be also given in the case of progression of an habit plane between a single martensite variant and the austenite :

$$\delta E_{ij}^{T} = -\frac{1}{V}\int_{S} \left[\varepsilon_{ij}^{T}\right] \delta\xi_n\, dS_n$$

where $\left[\varepsilon_{ij}^{T}\right] = \varepsilon_{ij}^{T+} - \varepsilon_{ij}^{T-}$ is the jump of the transformation strain if we cross the interface from the - side to the + side. Recalling that

$$\varepsilon_{ij}^{T+} - \varepsilon_{ij}^{T-} = -\frac{1}{2}(m_i\, n_j + m_j\, n_i)\, g = -R_{ij}\, g \tag{8}$$

we have also

$$\delta E_{ij}^{T} = \frac{1}{V}\int_{S} R_{ij}\, g\, \delta\xi_n\, n_n\, dS. \tag{9}$$

For an interface between two martensitic variants I and J with transformation-strain ε^{TI} and ε^{TJ} :

$$\delta E_{ij}^{T} = \frac{1}{V}\int_{S_{IJ}} \left(\varepsilon_{ij}^{TI} - \varepsilon_{ij}^{TJ}\right)\delta\xi_n\, n_n\, dS \tag{10}$$

or $$\delta E_{ij}^{T} = \frac{1}{V}\int_{S_{IJ}} \left(R_{ij}^{I} - R_{ij}^{J}\right) g\, \delta\xi_n\, n_n\, dS \tag{11}$$

The overall response due to phase transformation is clearly related to the transformation strain and the progression of the transformed volume by the movement of the interface. Given the evolution equation for

$\delta\xi_n$ as a function of a thermodynamical force, one can find the constitutive equation linking δE and $\delta\Sigma$, Σ, T and δT.
Since this thermodynamical driving force is strongly stress and temperature dependent, a "static" analysis of local stress fields at interfaces is reviewed in the next paragraph.

3. - Internal stress and stored energy associated with martensitic transformation

As for classical plasticity, the transformation strain field ε^T (r) possesses some incompatibilities leading to internal stresses and an associated stored energy.
These incompatibilities are of three types :

- the first one corresponds to the mechanism of variant formation itself
- the second corresponds to the interaction of martensitic plates with the microstructure of the austenite (dislocations, grains boundaries...)
- the last type are interactions inside the martensite itself (interaction between variants leading or not to an accomodated microstructure of martensite, non local interactions between martensite in several grains...).

The associated internal stress field plays a basic role on the developed microstructure and the overall behaviour. In the following, we recall some basic relations between the ε^T (r) field and the internal stress field.

Let ε^T (r) a given transformation strain field and ε^e (r) the elastic field necessarely associated to ε^T (r) in order to obtain a total compatible displacement field u_i whose gradient is

$$u_{(i,j)} (r) = \varepsilon^e_{ij} (r) + \varepsilon^T_{ij} (r) \tag{12}$$

The linear and homogeneous elastic behaviour is described by the constitutive equation

$$\sigma_{ij} (r) = C_{ijkl}\, \varepsilon^e_{kl} (r) \tag{13}$$

and equilibrium requires

$$\sigma_{ij,j} = 0. \tag{14}$$

The field equations (12) (14) together with the constitutive relation (13) yield the following equation for the total displacement field u_k (r)

$$C_{ijkl}\, u_{k,lj} - C_{ijkl}\, \varepsilon^T_{kl,j} = 0 \tag{15}$$

We assume that, at the boundary (∂V) of the (infinite) volume V, displacement is prescribed by

$$u^o_i = E_{ij}\, x_j \qquad (r \in \partial V) \tag{16}$$

The solution of (15) together with boundary condition (16) is obtained from the Green function G of the infinite medium with elastic constants C by

$$u_n (r) = u^o_n (r) - \int_V C_{ijkl}\, G_{jn} (r - r')\, \varepsilon^T_{kl,i} (r')\, dV' \tag{17}$$

After partial integration, derivation with respect to r and symmetrisation (m,n), one obtain for the total strain field ε (r)

$$\varepsilon_{nm} (r) = E_{nm} + \int_V C_{ijkl}\, \Gamma_{nmij} (r - r') \left(\varepsilon^T_{kl} (r') - \overline{\varepsilon}^T_{kl}\right) dV' \tag{18}$$

where $\Gamma_{mnij} = -\frac{1}{2} (G_{jn,im} + G_{jm,in})$
is the modified Green function introduced by Kröner [17].
We rewritte (18) after introduction of the deviation

$$\varepsilon'_{mn} (r) = \int_V C_{ijkl}\, \Gamma_{mnij} (r - r') \left(\varepsilon^T_{kl} (r') - \overline{\varepsilon}^T_{kl}\right) dV' \tag{19}$$

so that

$$\varepsilon_{mn}(r) = E_{mn} + \varepsilon'_{mn}(r) \tag{20}$$

with the property $\overline{\varepsilon'(r)} \equiv \frac{1}{V} \int_V \varepsilon'(r)\, dV = 0$

The internal stress field $\sigma(r)$ is obtained by

$$\sigma_{ij}(r) = C_{ijkl}\, \varepsilon^{e}_{kl}(r)$$

or

$$\sigma_{ij}(r) = C_{ijkl} \left(E_{kl} + \varepsilon'_{kl}(r) - \varepsilon^{T}_{kl}(r)\right) \tag{21}$$

and the overall stress Σ (= $\overline{\sigma}$) is deduced from (20)

$$\Sigma_{ij} = C_{ijkl} \left(E_{kl} - \overline{\varepsilon^{T}_{kl}}\right) \tag{22}$$

so that, the macroscopic part E^T of the total imposed strain E is equal to $\overline{\varepsilon^{T}}$.

The stored (or deformation strain) energy density w(r) of the field $\varepsilon^e(r)$ is only related to the transformation strain field $\varepsilon^T(r)$ and the boundary conditions. The total elastic-energy W for the volume V is obtained by

$$W = \frac{1}{2} \int_V \sigma_{ij}(r)\, \varepsilon^{e}_{ij}(r)\, dV \tag{23}$$

or from (12)

$$W = \frac{1}{2} \int_V \sigma_{ij}(r) \left(u_{i,j} - \varepsilon^{T}_{ij}(r)\right) dV \tag{24}$$

By partial integration and the conditions $u_i = E_{ij}\, x_j$ on ∂V equation, (24) is reduced to

$$W = \frac{1}{2} E_{ij}\, \Sigma_{ij}\, V - \frac{1}{2} \int_V \sigma_{ij}(r)\, \varepsilon^{T}_{ij}(r)\, dV \tag{25}$$

or, in terms of given quantities

$$W = \frac{1}{2} V\, E_{ij}\, C_{ijkl} \left(E_{kl} - E^{T}_{kl}\right) - \frac{1}{2} \int_V \sigma_{ij}(r)\, \varepsilon^{T}_{ij}(r)\, dV \tag{26}$$

If the field σ is decomposed into the macroscopic field Σ and internal field σ' (with the property $\overline{\sigma} = 0$), (26) can be written in the form

$$W = \frac{1}{2} V \left(E_{ij} - E^{T}_{ij}\right) C_{ijkl} \left(E_{kl} - E^{T}_{kl}\right) - \frac{1}{2} \int_V \sigma'_{ij}(r)\, \varepsilon^{T}_{ij}(r)\, dV \tag{27}$$

If Σ and ε^T are given, the expression of W is

$$W = \frac{1}{2} V\, \Sigma_{ij}\, C^{-1}_{ijkl}\, \Sigma_{kl} - \frac{1}{2} \int_V \sigma'_{ij}(r)\, \varepsilon^{T}_{ij}(r)\, dV \tag{28}$$

The σ' field is decomposed from (21) and (22) in both cases and may be written

if E, ε^T are given : $\sigma = C\,(E - E^T) + C\,(E^T + \varepsilon' - \varepsilon^T)$ (29)

if Σ, ε^T are given : $\sigma = \Sigma + C\,(E^T + \varepsilon' - \varepsilon^T)$ (30)

In most practical situations, it is useful to introduce the volume fraction f of martensite defined by $f = V_M/V$. In that case, we denote by $\overline{\varepsilon}^{TM} = \frac{1}{V_M} \int_{V_M} \varepsilon^T(r)\, dV$ the mean value of the transformation strain for the martensite.

Then E^T is simply related to $\overline{\varepsilon}^{TM}$ by $E^T = f\, \overline{\varepsilon}^{TM}$.

It is well known [17] that the operator Γ may be decomposed into a local part $\mathfrak{E}\, \delta(r-r')$ and a non local term $\mathfrak{F}$. If the non local part is neglected, equation (18) is reduced to the well known Eshelby [18] - Kröner [19] formula

$$\varepsilon(r) = E + S^E \left(\varepsilon^T(r) - f\, \overline{\varepsilon}^{TM}\right) \tag{31}$$

where S^E is the Eshelby tensor.

So that the ε' tensor is now

$$\varepsilon'_{ij}(r) = S^E_{ijkl} \left(\varepsilon^T_{kl}(r) - f\, \overline{\varepsilon}^{TM}_{kl}\right) \tag{32}$$

and

$$\sigma'_{ij}(r) = C_{ijkl} \left(I_{klmn} - S^E_{klmn}\right) \left(f\, \overline{\varepsilon}^{TM}_{mn} - \varepsilon^T_{mn}(r)\right) \tag{33}$$

The strain energy (28) is obtained by

$$W = \frac{1}{2} V \left(E_{ij} - f\, \overline{\varepsilon}^{TM}_{ij}\right) C_{ijkl} \left(E_{kl} - f\, \overline{\varepsilon}^{TM}_{kl}\right) - \frac{1}{2} C_{ijkl} \left(I_{klmn} - S^E_{klmn}\right) (fV) \left(f\, \overline{\varepsilon}^{TM}_{mn} \overline{\varepsilon}^{TM}_{ij} - \overline{\varepsilon^T_{mn} \varepsilon^T_{ij}}^{M}\right) \tag{34}$$

where $\overline{\varepsilon^T_{mn} \varepsilon^T_{ij}}^{M} = \frac{1}{V_M} \int_{V_M} \varepsilon^T_{ij}(r)\, \varepsilon^T_{mn}(r)\, dV$

Equation (18) solves completely the problem of internal stress associated with an incompatible strain field $\varepsilon^T(r)$ and particularly the discontinuity conditions at interfaces.

More practical and useful relations at interfaces may be obtained directly from the field equation (12) and (14) without using the Green function. These relations should be used directly in the expression for the driving force introduced in the next part.

Let ε^+_{ij} and ε^-_{ij} $\left(\text{resp. } \varepsilon^{T+}_{ij} \text{ and } \varepsilon^{T-}_{ij}\right)$ be the total strain (resp. transformation strain) at the interface with unit normal n oriented from the -part to the +part between two media with the same linear elastic behaviour described by tensor C. σ^+_{ij} and σ^-_{ij} are the stresses at the interface and they may be due to applied, local and non local internal stress.

At the interface, the particular form of the field equations (12) and (14) are

$$\varepsilon^+_{ij} - \varepsilon^-_{ij} = \frac{1}{2} \left(\lambda_i n_j + \lambda_j n_i\right) \tag{35}$$

and

$$\sigma^+_{ij} n_j - \sigma^-_{ij} n_j = 0 \tag{36}$$

where λ is an arbitrary vector function on the interface.

Stress and strain are related by

$$\sigma^{+}_{ij} = C_{ijkl}\left(\varepsilon^{+}_{kl} - \varepsilon^{T+}_{kl}\right)$$

$$\sigma^{-}_{ij} = C_{ijkl}\left(\varepsilon^{-}_{kl} - \varepsilon^{T-}_{kl}\right) \tag{37}$$

Introducing (37) into (36) gives

$$n_j\, C_{ijkl}\left(\varepsilon^{-}_{kl} - \varepsilon^{T-}_{kl}\right) = \sigma^{+}_{ij}\, n_j \tag{38}$$

and, by (35),

$$n_j\, C_{ijkl}\left(\varepsilon^{+}_{kl} + \frac{1}{2}\left(\lambda_k\, n_l + \lambda_l\, n_k\right) - \varepsilon^{T-}_{kl}\right) = \sigma^{+}_{ij}\, n_j \tag{39}$$

so that the unknown vector λ_k is obtained by

$$c_{ik}\, \lambda_k = n_j\, C_{ijmn}\left(\varepsilon^{T+}_{mn} - \varepsilon^{T-}_{mn}\right) \tag{40}$$

where $c_{ik} = C_{ijkl}\, n_j\, n_l$ is the Christoffel matrix.
Now, from (38), (39) and (35) it is easy to see that ε^{+} and σ^{+} may be obtained explicitely from the values of ε^{-} and σ^{-} only.
For isotropic elasticity, with Lame constants λ and μ

$$C_{ijkl} = \lambda\, \delta_{ij}\, \delta_{kl} + \mu\, (\delta_{ik}\, \delta_{jl} + \delta_{il}\, \delta_{jk}) \tag{41}$$

(38), (35) are solved by

$$\sigma^{+}_{ij} - \sigma^{-}_{ij} = B_{ijkl}\left(\varepsilon^{T-}_{kl} - \varepsilon^{T+}_{kl}\right) \tag{42}$$

where B is given by Hill [20] or Eshelby [18]

$$B_{ijkl} = 2\,\mu\left(F_{ijkl} + \frac{\lambda}{\lambda + 2\,\mu}\,(\delta_{ij} - n_i\, n_j)\,(\delta_{kl} - n_k\, n_l)\right)$$

and $$F_{ijkl} = \frac{1}{2}\,(\delta_{ik} - n_i\, n_k)\,(\delta_{jl} - n_j\, n_l) + \frac{1}{2}\,(\delta_{jk} - n_j\, n_k)\,(\delta_{il} - n_i\, n_l)$$

Formula (42) may be used in order to deduce σ^{+} from σ^{-} which is obtained from the field equations (18) and (21).
From the general definition of the martensitic transformation, it appears that (42) corresponds also to the small strain formulation of the Wechsler, Liebermann and Read [21] theory.
Assuming that in the austenite $\varepsilon^{T+} = 0$ and that inside the martensite ε^{T-} is composed of a given Bain strain ε^{B} and a lattice invariant strain defined by $\varepsilon^{LI}(\gamma)$, the condition of inhomogeneous strain without stress reduces to

$$B_{ijkl}\left(\varepsilon^{B}_{kl} + \varepsilon^{LI}_{kl}(\gamma)\right) = 0$$

which has a non trivial solution if $\mathrm{Det}\left(\varepsilon^{B}_{kl} + \varepsilon^{LI}_{kl}(\gamma)\right) = 0.$

Equation (42) may also be used for the determination of the self accomodating character of variants.

4. - Thermodynamic force on interfaces

The rate at which the total energy of the system Austenite + Martensite + Loading system varies if the configuration of the system changes by (a virtual) interface movement defines the thermodynamical driving force for this evolution. Eshelby [16] introduced in solid mechanics the elastic energy-momentum tensor which is a useful tool for this calculation.

The total potential energy ϕ of the system is composed of

- an elastic strain energy with density w (r)
- the potential energy of the loading mechanism due to a density of given surface forces $T_i^d = \Sigma_{ij} n_j$ on ∂V
- the crystallographic free energy, usually called chemical free energy density φ (r)
- the interfacial energy of the interface between all the constituents. This last term is generally neglected in thermoelastic phase transformation.

If the other sources of energy are neglected, ϕ is written as

$$\phi = \int_V w(r)\, dV - \int_S T_i^d u_i\, dS + \int_V \varphi(r)\, dV \tag{43}$$

w takes into account both the stress due to applied surface forces T_i^d and internal stresses associated with the incompatibilities of the ε^T (r) field and some other (dislocation for example) inelastic field.

Let us first analyse the variation of the last term. φ (r) can be assumed to be piece wise uniform, equal to φ_A in the austenitic part and φ_M in the martensite independently of the orientation of the variant. With the same notation as in § 2, $\int_V \varphi(r)\, dV$ is written as

$$\int_V \varphi(r)\, dV = \int_V \varphi_A + (\varphi_M - \varphi_A) \sum_I \theta^I(r)\, dV \tag{44}$$

The variation $\delta \int_V \varphi(r)\, dV$ is obtained for a martensitic inclusion by

$$\delta \int_V \varphi(r)\, dV = + \int_{S_I} (\varphi_M - \varphi_A)\, \delta\xi_n\, n_n\, dS \tag{45}$$

or if S denote all the internal interfaces

$$\delta \int_V \varphi(r)\, dV = + \int_S (\varphi_M - \varphi_A)\, \delta\xi_n\, n_n\, dS = - \int_S [\varphi]\, \delta\xi_n\, n_n\, dS \tag{46}$$

The variation of the two first terms are obtained by Eshelby [16] througth the energy-momentum tensor in the form

$$\delta \left[\int_V w(r)\, dV - \int_{S_{ext}} T_i^d f_i\, dS \right] = - \int_S [P_{lj}]\, \delta\xi_l\, n_j\, dS \tag{47}$$

where $[P_{lj}] = [w]\, \delta_{lj} - \sigma_{ij}\, [u_{i,l}]$ (48)

is the energy momentum tensor.

Assuming uniform elasticity, [w] can be expressed by

$$[w] = \frac{1}{2} \left(\sigma_{ij}^{+} \left(\varepsilon_{ij}^{+} - \varepsilon_{ij}^{T+} \right) - \sigma_{ij}^{-} \left(\varepsilon_{ij}^{-} - \varepsilon_{ij}^{T-} \right) \right) \tag{49}$$

or

$$[w] = \frac{1}{2} \left(\left(\varepsilon_{ij}^{+} - \varepsilon_{ij}^{T+} \right) S_{ijkl} \left(\varepsilon_{kl}^{+} - \varepsilon_{kl}^{T+} \right) - \left(\varepsilon_{ij}^{-} - \varepsilon_{ij}^{T-} \right) S_{ijkl} \left(\varepsilon_{kl}^{-} - \varepsilon_{kl}^{T-} \right) \right) \tag{50}$$

Due to the symmetry of the S tensor, (50) is transformed in

$$[w] = \frac{1}{2}\left(\sigma^{+}_{ij} + \sigma^{-}_{ij}\right)\left[\varepsilon_{ij} - \varepsilon^{T}_{ij}\right] \tag{51}$$

The term $\sigma_{ij}\,[u_{i,l}]\,n_j$ is also equal to $\sigma_{ij}\,[u_{i,j})\,n_l$ from compatibility conditions (35) and (36) along the interface so that

$$\sigma_{ij}\,[u_{i,l}]\,n_j = \sigma_{ij}\,[\varepsilon_{ij}]\,n_l \tag{52}$$

From (52) and (51), one obtains for $[P_{lj}]\,n_j$

$$[P_{lj}]\,n_j = -\frac{1}{2}\left(\sigma^{+}_{ij} + \sigma^{-}_{ij}\right)\left[\varepsilon^{T}_{ij}\right] n_l \tag{53}$$

as obtained also by Hill [22].
Using (46) and (53), the variation of the total energy of the system is written as

$$\delta\phi = -\int_S \left\{-\frac{1}{2}\left(\sigma^{+}_{ij} + \sigma^{-}_{ij}\right)\left[\varepsilon^{T}_{ij}\right] + [\varphi]\right\} n_l\,\delta\xi_l\,dS \tag{54}$$

so that the force F_l on the interface, defined by

$$\delta\phi = -\int_S F_l\,\delta\xi_l\,dS \tag{55}$$

is equal to

$$F_l = -\frac{1}{2}\left(\sigma^{+}_{ij} + \sigma^{-}_{ij}\right)\left[\varepsilon^{T}_{ij}\right] n_l + [\varphi]\,n_l \tag{56}$$

Equation (56) may be particularized for several special cases.

a) The first one corresponds to the relaxed habit plane of a single variant between martensite and austenite. Classical plasticity is neglected so that

$$[\varepsilon^{T}] = -\varepsilon^{TI}_{ij} = -R^{I}_{ij}\,g$$

where R^{I}_{ij} is the orientation tensor of the transformation strain of the variant I and g the amplitude of transformation shear.
Since this situation corresponds to vanishing internal stress

$$\frac{1}{2}\left(\sigma^{+}_{ij} + \sigma^{-}_{ij}\right) = \Sigma_{ij}$$

where Σ is the applied stress (associated with T^{d}_{i}).

For a low range of temperature variation, the change in chemical free energy $\varphi_M - \varphi_A$ is usualy written as $B\,(T - T_o)$.
The force on the interface is now reduced to

$$F_l = \left(\Sigma_{ij}\,R^{I}_{ij}\,g - B\,(T - T_o)\right) n_l$$

which corresponds to the result of Patoor and all [14] or for the uniaxial case to the criterion of Patel and Cohen [23].

b) The second case corresponds to the part of the interface between two compatible variants with transformation strain $\varepsilon^{TI}_{ij} = R^{I}_{ij}\,g$ and $\varepsilon^{TJ}_{ij} = R^{J}_{ij}\,g$.

In that case there is no driving force associated with the chemical energy and, by compatibility of the transformation strain field, one has

$$F_l = \Sigma_{ij}\left(R^I_{ij} - R^J_{ij}\right) g\, n_l$$

where n is oriented from I to J.
Only applied stress can move the interface between two compatible variants leading to the reorientation effects.

5. - Conclusions

For perfect phase transformations (no dissipation by dry or viscous friction), equations (21), (42) and the condition $F_l = 0$ locally for each interface solves completely the problem.
In case of shape memory alloys with hysteresis, the condition $F_l = 0$ must be replaced by the inequations $F_l \leq F_c$ where F_c is a critical force which must be reached for interface movement.
In the case of non linear viscous friction, a phenomenological evolution equation for $\frac{\delta \xi_l}{\delta t}$ may be written in the form of a power law like

$$\frac{\delta \xi_l}{\delta t} = K\,(F)^m\, n_l \qquad (57)$$

where F is the norm of F_l.
The macroscopic transformation strain rate is then obtained by (7) and (57)

$$\frac{\delta E^T_{ij}}{\delta t} = \frac{1}{V}\sum_I \int_{S_I} \varepsilon^{T_I}_{ij}\, K\,(F)^m\, dS \qquad (58)$$

Results based on this approach are in progress in this case but also for other inelastic properties of metals.

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